$$Q_1 = (kb)\left\{\frac{\log p}{1-p^2} - \frac{1+p^{-2}}{4}\right\}$$

$$R_1 = kb + P_1$$

$$S_1 = \frac{1}{kb} + Q_1$$

for $n \geq 2$

$$E_r^{II} = -j\frac{E^{II}}{2n}\left\{\frac{\rho^{n-1}}{(ka)^n}(P_n - \rho^2 Q_n) + \frac{(ka)^n}{\rho^{n+1}}(R_n - \rho^2 S_n)\right\}\cos n\theta$$

$$E_\theta^{II} = j\frac{E^{II}}{2n}\left\{\frac{\rho^{n-1}}{(ka)^n}(P_n + \rho^2 Q_n) - \frac{(ka)^n}{\rho^{n+1}}(R_n + \rho^2 S_n)\right\}\sin n\theta$$

$$Z_0 H_r^{II} = -j\frac{E^{II}}{2n}\left\{\frac{\rho^{n-1}}{(ka)^n}(P_n + 2n\rho^{-n} + \rho^2 Q_n) - \frac{(ka)^n}{\rho^{n+1}}(R_n + 2n\rho^n + \rho^2 S_n)\right\}\sin n\theta$$

$$Z_0 H_\theta^{II} = -j\frac{E^{II}}{2n}\left\{\frac{\rho^{n-1}}{(ka)^n}(P_n + 2n\rho^{-n} - \rho^2 Q_n) + \frac{(ka)^n}{\rho^{n+1}}(R_n + 2n\rho^n - \rho^2 S_n)\right\}\cos n\theta$$

with

$$P_n = \frac{1}{p^{-n} - p^n}\left\{(ka)^2\left(\frac{1}{n-1} - \frac{p^{-2n}}{n+1}\right) - \frac{2(kb)^2}{n^2-1}\right\}$$

$$R_n = \frac{1}{p^{-n} - p^n}\left\{(ka)^2\left(\frac{p^{2n}}{n-1} - \frac{1}{n+1}\right) - \frac{2(kb)^2}{n^2-1}\right\}$$

$$Q_n = \frac{p^{-n}}{n+1} \qquad S_n = \frac{p^n}{n-1}.$$

References

[1] J. R. Pierce and L. M. Field, "Traveling-wave tubes," *Proc. IRE*, vol. 35, pp. 108–111, Feb. 1947.

[2] S. Sensiper, "Electromagnetic wave propagation on helical conductors," M.I.T. Res. Lab. Electron., Cambridge, Mass., Rep. 194, May 1951.

[3] J. Falnes, "Hybrid modes of a circular cylindrical cavity containing a unidirectionally conducting coaxial sheet," in *Electromagnetic Theory and Antennas*, pt. 2, E. C. Jordan, Ed. New York: Pergamon, 1962, pp. 941–950.

[4] G. Goudet and P. Chavance, *Ondes Centimétriques*, 1955, p. 73.

[5] Y. Garault, "Hybrid EH guided waves," in *Advances in Microwaves*, vol. 5, L. Young, Ed. New York: Academic, 1970.

[6] Y. Garault and J. P. Fenelon, "Hybrid waves in the through periodic waveguide," *C. R. Acad. Sci.*, vol. 567, pp. 540–543, 1968.

[7] H. Hahn, "The deflecting mode in the circular iris-loaded waveguide of an RF particle separator," Brookhaven Nat. Lab., Brookhaven, N.Y., BNL-AGS Internal Rep. HH-5, 1962.

[8] H. Hahn and H. J. Halama, "Mode identification in the iris-loaded waveguide of an RF particle separator," Brookhaven Nat. Lab., Brookhaven, N.Y., BNL-AGS Internal Rep. HH/HJH-2, 1963.

[9] C. Fray, "La ligne à anneaux blindée," Thèse de 3 ème cycle, Faculty Sciences, Univ. Poitiers, Poitiers, France, May 1970.

[10] J. R. Pierce and P. K. Tien, "Coupling of modes in helixes," *Proc. IRE*, vol. 42, pp. 1389–1396, Sept. 1954.

[11] V. Vaghin, "Etude des ondes électromagnétiques hybrides dans les structures cylindriques. Application des résultats aux modes déflecteurs," CERN Rep. 71-4.

[12] P. H. Bernard, H. Lengeler, and V. Vaghin, "New disk-loaded waveguide for the CERN separator," CERN Rep. 70-26, Sept. 17, 1970.

Resonance Conditions of Open Resonators at Microwave Frequencies

TATSUO ITOH, MEMBER, IEEE, AND RAJ MITTRA, FELLOW, IEEE

Abstract—**This paper presents an extension of Vajnshtejn's approach for computing the resonance frequencies and loss factors of Fabry–Perot (FP) resonators at microwave frequencies. Numerical results are presented for FP resonators operated at microwave through millimeter frequency range.**

Manuscript received March 28, 1973; revised September 14, 1973. This work was supported in part by the Dikewood Corporation Subcontract DC-SC-72-19 for the Air Force Special Weapons Center, Prime Contract Number F29601-72-C-0087, and in part by the U.S. Army Research Grant DA-ARO-D-31-124-G77.

The authors are with the Department of Electrical Engineering, University of Illinois, Urbana, Ill. 61801.

I. Introduction

FABRY–PEROT (FP) and other types of open resonators find useful applications at optical as well as millimeter or microwave frequencies. Typically, these resonators consist of two plane or curved mirrors facing each other. Though much has been reported on the analysis of such resonators, most of these analyses employ a conventional integral equation approach [1], [2]. An alternate and efficient method for attacking this problem has been introduced by Vajnshtejn [3] by regarding the resonator structure as a truncated parallel-plate waveguide. He begins by computing the reflection properties at the open ends of the waveguide (open side walls of the resonator) and employs a simple transmission-line theory for deriving the resonance condition. In computing the reflection coefficient he makes advantageous use of the asymptotic forms which are valid at optical frequencies. Other workers, such as Li and Zucker [4], have also found this approach useful for solving open resonator problems.

The purpose of this paper is to extend Vajnshtejn's approach to the microwave frequency range where the optical approximation is no longer valid. This is done by working with a more exact form of the expression for the reflection coefficient which is valid for arbitrary frequencies.

The readers who are interested only in numerical computation may bypass the theories in Section II of this paper and directly follow the numerical procedures listed in Section III.

II. Derivation of the Eigenvalue Equation

Fig. 1 shows the cross section of the plane-mirror open resonator. For simplicity of analysis, it is assumed that the resonator is infinite and uniform in the y direction. We restrict ourselves to the case of TM (with respect to z) fields, although the TE case can be handled in a similar manner.

We will first describe the formula for computing the reflection coefficient at the open end of the resonator. This quantity is necessary in deriving the eigenvalue equation of the resonance characteristics. To this end, the resonator is viewed as a parallel-plate waveguide (infinite y dimension) in which the field is traveling in the $\pm z$ direction. If we assume that there is a negligible amount of coupling between the two open ends at $z = 0$ and $z = -l$, it is possible to express the reflection coefficient at one of the open ends, say, at $z = 0$, via the Wiener–Hopf procedure [5].

Assume that the TM_{q0} mode is incident at $z = 0$ from the left. The field inside the semi-infinite parallel plates is

$$H_y = \cos\left[\frac{q\pi}{2b}(x-b)\right]\exp(i\beta_q z) + \sum_{n=0}^{\infty} R_{nq}\cos\left[\frac{n\pi}{2b}(x-b)\right]\exp(-i\beta_n z) \quad (1a)$$

$$\beta_n = \left[k^2 - \left(\frac{n\pi}{2b}\right)^2\right]^{1/2} = i\left[\left(\frac{n\pi}{2b}\right)^2 - k^2\right]^{1/2} \quad (1b)$$

$$k = \omega/c \quad (1c)$$

where ω is the angular frequency, c is the velocity of light,

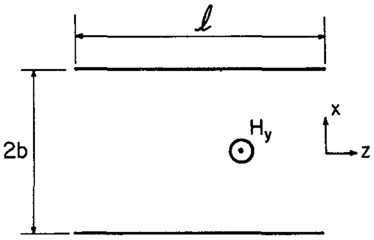

Fig. 1. Plane-mirror open resonator.

and R_{nq} is the reflection coefficient of the TM_{n0} mode due to the TM_{q0} mode incidence. The time factor $\exp(-i\omega t)$ will be suppressed throughout this paper. We will also restrict ourselves to the case when q is even and hence n is even, though the odd mode case can be handled in much the same way.

The reflection coefficients R_{nq} can be derived once the field expression for H_y is available. The latter may be obtained via the application of the Wiener–Hopf procedure. Since the details of derivation and the solution of the Wiener–Hopf equation have appeared in a number of previous publications [5], [6], we will omit the details and quote only the final expression for the reflections coefficients R_{nq}:

$$R_{nq} = -\frac{(\beta_q + k)(\beta_n + k)}{(\beta_q + \beta_n)\beta_n} G_+(\beta_n) G_+(\beta_q) \quad (2)$$

where $G_+(\alpha)$ is the so-called factorized function of the Wiener–Hopf kernel and is typically expressed in terms of an expression containing an infinite product [5]. The expression for $G_+(\alpha)$ is

$$G_+(\alpha) = \left(\frac{\sin kb}{kb}\right)^{1/2} \exp\left\{\frac{i\alpha b}{\pi}\left[1 - C + \ln\left(\frac{2\pi}{kb}\right) + i\frac{\pi}{2}\right]\right\} \cdot \exp\left[\frac{ib(\alpha^2 - k^2)^{1/2}}{\pi}\ln\left(\frac{\alpha - (\alpha^2 - k^2)^{1/2}}{k}\right)\right] \cdot \prod_{n=2,\text{even}}^{\infty}\left(1 + \frac{\alpha}{\beta_n}\right)\exp\left(i\frac{2\alpha b}{n\pi}\right) \quad (3)$$

where $C = 0.57721\cdots$ (Euler's constant).

It is found, however, that the computation of $G_+(\alpha)$ is quite laborious for large kb, owing to the slow convergence of the infinite product. The situation becomes especially critical when kb is in the optical or quasi-optical range. To alleviate this difficulty, it is useful to use an alternate form for $G_+(\alpha)$ which converges much more rapidly:

$$G_+(\alpha) = \left(\frac{\sin kb}{kb}\right)^{1/2} \exp\left(\frac{ikb}{2}\right)\left(1 + \frac{\alpha}{k}\right)^{-1/2} \cdot \exp\left\{\frac{i}{2\pi}\int_P \ln\left[1 + \frac{2\alpha b}{[s(s - i4kb)]^{1/2}}\right] \cdot \frac{\exp(i2kb)\exp(-s)}{\exp(i2kb)\exp(-s) - 1}\, ds\right\}. \quad (4)$$

The details of the transformation from (3) to (4) may be found in Bates [6]. The integration path P in (4) is along both sides of the branch cut located on the positive real axis in the complex s plane. Because of the factor $\exp(-s)$, the integral converges quite rapidly for any value of kb. It may be verified that the asymptotic form of G_+ used by Vajnshtejn [3] can be derived by taking the limit of $kb \to \infty$ in (4).

The next step is the derivation of the eigenvalue equation for the resonance condition of the TM_{q0m} mode where m is the resonant mode index associated with the field variation in the z direction corresponding to the TM_{q0} waveguide mode. For the resonant TM_{q0m} mode in a cavity, the standing wave fields may be written in a standard form as shown below. Depending on whether the H_y field is even (m even) or odd (m odd) with respect to $z = -l/2$, one writes the H_y field as

$$H_y = \begin{cases} K \cos\left[\frac{q\pi}{2b}(x-b)\right] \cos\left[\beta_q\left(z+\frac{l}{2}\right)\right], & \text{even} \quad (5a) \\ K \cos\left[\frac{q\pi}{2b}(x-b)\right] \sin\left[\beta_q\left(z+\frac{l}{2}\right)\right], & \text{odd.} \quad (5b) \end{cases}$$

For convenience of later comparison, we rewrite (5) as

$$H_y = \begin{cases} \bar{K} \cos\left[\frac{q\pi}{2b}(x-b)\right] \{\exp(i\beta_q z) \\ \quad + \exp(-i\beta_q l)\exp(-i\beta_q z)\}, & \text{even} \quad (6a) \\ \bar{K} \cos\left[\frac{q\pi}{2b}(x-b)\right] \{\exp(i\beta_q z) \\ \quad - \exp(-i\beta_q l)\exp(-i\beta_q z)\}, & \text{odd} \quad (6b) \end{cases}$$

where K and $\bar{K}$ in (5) and (6) are arbitrary constants. Returning to (1a) we note that the TM_{q0} field in the open-ended waveguide have the form

$$H_y = \cos\left[\frac{q\pi}{2b}(x-b)\right] \{\exp(i\beta_q z) + R_{qq}\exp(-i\beta_q z)\}. \tag{7}$$

The resonance condition may now be obtained by comparing (6) and (7). We note that the open-ended waveguide satisfies the resonance condition if we set

$$R_{qq} = (-1)^m \exp(-i\beta_q l), \qquad m = 0, 1, 2, \cdots \tag{8}$$

which is the desired characteristic equation for the open resonator. The unknown here is β_q which in turn determines the resonant frequency ω_c. It should be pointed out that since the resonator is open, (8) has solutions only for complex values of angular frequency ω_c, and hence of wavenumber k_c. The imaginary part of ω_c (or k_c) accounts for the spill-over or radiation losses at the open end of the resonator.

III. Numerical Algorithm

Although a closed-form solution of the nonlinear (8) is not possible, it is nevertheless tractable via the use of iterative algorithms. To this end, let us first express R_{qq} as

$$R_{qq} = -\exp[i\beta_q(s_1 + is_2)]. \tag{9}$$

Equation (9) may be interpreted as follows. If the open end of the parallel-plate waveguide was an ideal open circuit for the TM_{q0} mode, R_{qq} would be exactly -1, s_1,s_2 would be identically zero, and the resonant frequencies would be purely real. However, in the actual situation s_1 and s_2 are not zero; the value of s_1 accounts for the additional phase shift, while a nonzero s_2 represents the presence of radiation or spill-over losses.

Substituting (9) into (8), one obtains

$$\exp[i\beta_q(l + s_1 + is_2)] = (-1)^{m+1}, \quad m = 0,1,2,\cdots \tag{10}$$

when m is the resonance index associated with the field variation in the z direction. Solving for β_q we get

$$\beta_q = \frac{(m+1)\pi}{l + s_1 + is_2}. \tag{11}$$

The complex resonance frequency ω_c, can be determined from (1b), (1c), and (11). The pertinent equations are

$$\omega_c = ck_c$$

$$k_c^2 = \left(\frac{q\pi}{2b}\right)^2 + \beta_q^2 = \left(\frac{q\pi}{2b}\right)^2 + \left[\frac{(m+1)\pi}{l + s_1 + is_2}\right]^2. \tag{12}$$

The real and imaginary parts of k_c are obtained from (12) and are given by the expressions

$$k_c = k_1 - ik_2 \tag{13a}$$

$$k_1 = \frac{1}{\sqrt{2}}[X + (X^2 + 4Y^2)^{1/2}]^{1/2} \tag{13b}$$

$$k_2 = Y/k_1 \tag{13c}$$

where

$$X = \left(\frac{q\pi}{2b}\right)^2 + \frac{(m+1)^2\pi^2[(l+s_1)^2 - s_2^2]}{[(l+s_1)^2 - s_2^2]^2 + 4s_2^2(l+s_1)^2} \tag{14a}$$

$$Y = \frac{(m+1)^2\pi^2 s_2(l+s_1)}{[(l+s_1)^2 - s_2^2]^2 + 4s_2^2(l+s_1)^2}. \tag{14b}$$

The numerical routine for finding the resonance condition of the TM_{q0m} mode is as follows.

1) For a given set of parameters b,l,q,m, let $s_1 = s_2 = 0$ and find k_c from (13) and (14).

2) Find the value of R_{qq} from (4) and (2) with $k = k_c$.

3) Substitute the resulting value into (9) and find new values of s_1 and s_2.

4) Find the new value of k_c from (13) and (14) and repeat steps 2) and 3) until the process converges to satisfy the stopping criterion.

5) The resonance condition is given by the value of k_c expressed in terms of the final values of s_1 and s_2.

As the stopping criterion, the values of s_1, and s_2 and also the values of R_{qq} in the nth and $(n+1)$th iterations, are compared. If all of these differences are smaller than 10^{-4}, the iteration process is terminated and k_1 and k_2 are derived from (13).

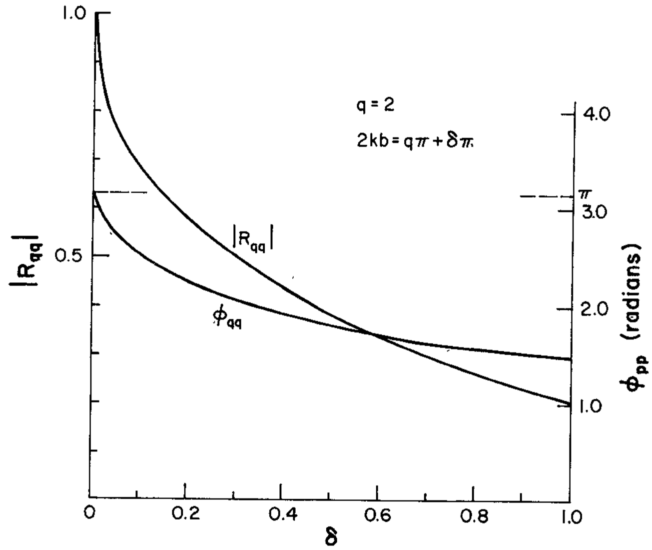

Fig. 2. Reflection coefficient at the open end of a semi-infinite waveguide.

TABLE I

m	This method k_1	This method k_2	Vajnshtejn k_1	Vajnshtejn k_2
0	157.0807	0.893×10^{-4}	157.0808	0.922×10^{-4}
1	157.0842	3.58×10^{-4}	157.0841	3.69×10^{-4}
2	157.0900	8.07×10^{-4}	157.0900	8.30×10^{-4}

$b = 5$ cm, $\ell = 5$ cm, $q = 500$

Units of k_1 and k_2 are cm^{-1}.

IV. Numerical Results

Fig. 2 shows the typical plot of R_{qq} for the $q = 2$ case, where $\delta = (2kb/\pi) - q$ and $R_{qq} = |R_{qq}| \exp(-i\phi_{qq})$. As expected R_{qq} becomes -1 at the cutoff $\delta = 0$. These curves are identical to the ones found in [6]. To check the accuracy of the present method, several cases of large q, e.g., $q = 500$, were computed and compared with the resonance condition derived from Vajnshtejn's asymptotic formula. As is evident from Table I, the two results agree quite well.

TABLE II

q	m	f_1 (GHz)	Resonator Q	
2	0	15.23	254.24	
	1	15.91	67.24	
	2	17.00	32.51	
10	0	75.05	8020	
	1	75.21	2005	($\lambda \simeq 4$ mm)
	2	75.48	891	
50	0	375.01	3.56×10^5	
	1	375.05	0.89×10^5	($\lambda \simeq 0.8$ mm)
	2	375.10	0.39×10^5	

$b = 1$ cm, $\ell = 5$ cm

Table II shows some of the examples of resonant frequency $f_1 = (ck_1/2\pi)$ and the quality factor $Q = (k_1/2k_2)$ of the open resonator at microwave through millimeter frequencies. As expected, the loss due to the diffraction (spill-over loss) increases as the transverse mode number m is increased for the same q. It is also clear that the loss is smaller for larger values of q.

V. Conclusions

A numerically efficient method is presented for computing resonance conditions of an open resonator of the FP type. The number of iterations required is usually less than 10 (typically as small as 4). Typical computation time on the IBM 360/75 is about 2 s.

References

[1] A. G. Fox and T. Li, "Resonant modes in maser interferometer," *Bell Syst. Tech. J.*, vol. 40, pp. 453–488, Mar. 1961.
[2] G. D. Boyd and J. P. Gordon, "Confocal multimode resonator for millimeter through optical wavelength masers," *Bell Syst. Tech. J.*, vol. 40, pp. 489–508, Mar. 1961.
[3] L. A. Vajnshtejn, "Open resonators for lasers," *Sov. Phys.—JETP*, vol. 17, pp. 709–719, Sept. 1963.
[4] T. Li and H. Zucker, "Modes of a Fabry-Perot laser resonator with output-coupling apertures," *J. Opt. Soc. Amer.*, vol. 57, pp. 984–986, Aug. 1967.
[5] R. Mittra and S. W. Lee, *Analytical Techniques in the Theory of Guided Waves*. New York: Macmillan, 1971.
[6] C. P. Bates, "Internal reflections at the open end of a semi-infinite waveguide," *IEEE Trans. Antennas Propagat.*, vol. AP-18, pp. 230–235, Mar. 1970.